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Forms for Probability Ascriptions

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I apply the distinctions between the probability of a conditional, and a conditional with a probabilistic consequent, to quantum theory. I concentrate on an application hardly studied in the literature: namely, the case where the antecedent of the conditional states which quantity is measured, and the consequent states which value the quantity has. I show how we can construe quantum theory as providing propositions of these kinds, both for intrinsic possessed values, and for measurement results. I also show that most construals satisfy a plausible constraint requiring a kind of independence between which quantity is measured and what the value or result is.

1. PROBABILITIES AND CONDITIONALS: THE PROJECT

In this paper, I apply philosophical distinctions between various ways of combining probabilities and conditionals to quantum theory. More specifically, I consider the distinctions, established in recent philosophy of language, between the following three kinds of proposition, where X and Y are some propositions, $\rightarrow$ is a conditional connective, pr is some probability function, and z is a number:

(ProbCond) $\mathrm{pr}(X \rightarrow Y) = z$ (probability of a conditional)
(ProbCons) $X \rightarrow (\mathrm{pr}(Y) = z)$ (conditional with probabilistic consequent)
(CondProb) $\mathrm{pr}(Y/X) = z$ (conditional probability)

These need to be distinguished in the sense that they can take different truth-values. The distinction between (ProbCond) and (CondProb) is well recognized, following Lewis (1986, Chapter 20). As to the two distinctions involving (ProbCons), there is no single seminal paper. But various authors have noted one or other distinction: Lewis (1986, pp. 175–179, 331–333)

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distinguishes (ProbCons) from (CondProb), and Skyrms (1980, p. 138; 1984, p. 103) distinguishes it from (ProbCond).

The distinction between (ProbCond) and (CondProb) has already been applied to the state spaces of classical and quantum physics, for the case where both X and Y ascribe a value to a physical quantity, and thus correspond to subsets of the state space (a subset of phase space, or a subspace of Hilbert space). Indeed, for quantum physics, there is a considerable quantum logical literature relating the conditional and transition probability. [See, for example, Bugajski (1978) and Butterfield (1987), which both treat classical and quantum physics equally].

But the distinctions have hardly been applied to a physical theory's probability ascriptions, i.e., statements of the probability either of a measurement result or of an intrinsic possessed value, conditional on a specification of which quantity is supposed to be measured. This is the case I will consider. So for me, X states that a measurement of a certain quantity is performed, and Y states a corresponding measurement result or intrinsic value. (For short, I will say "result" and "value.") And the "pr" will be provided by the theory: for quantum theory, by the state vector or mixed state.

In fact we can construe both phase space and Hilbert space as providing propositions of the first two kinds, for both results and values. To consider two kinds of proposition, for both classical and quantum physics, and for results and values, one has $2 \times 2 \times 2 = 8$ cases. Each case involves a formal construction: one uses the state space to build a set of possible worlds that (i) includes as propositions (sets of worlds) the specification of which quantity is measured, and (ii) supports a conditional connective $\rightarrow$ as a binary operation on sets of worlds, and (iii) is also a probability space.

For reasons of space, I have to be brief. So I will set aside phase space (details available on request), and I will present the Hilbert space constructions informally. But little will be lost: the constructions are simple and similar in the phase space and Hilbert space cases, and the technicalities I omit (e.g., definitions of σ-fields) are standard and easily filled in. But although the constructions are technically simple, they establish four conceptual or philosophical points. The main one will be that in most of the eight cases the constructions can be done so as to satisfy constraints that reflect the distinction between results and values. But I will postpone these constraints until Section 3, and spend the rest of this section describing the other three points. This will also set my project in context.

The first point concerns the fact that I do not treat (CondProb). This may seem strange because (CondProb) is the most familiar of the three kinds, especially within physics; and because it seems the simplest, since it involves no $\rightarrow$. In fact it is technically difficult, once we set aside the special case of considering only finitely many quantities (for which, it is indeed simple—again, I must omit details). Thus in both phase space and Hilbert space, there are uncountably many quantities; and so there are uncountably many specifications of what quantity or quantities are to be measured (uncountably many whether or not some combinations of quantities are not comeasurable). (CondProb) brings such specifications into the algebra of events, i.e., the domain of pr. And since not all can be assigned nonzero probability, we face the problem of how to make sense of conditional probability in such a case. [As we will see, this problem does not arise for (ProbCond) and (ProbCons), because specifications of what is measured, though of course in the algebra of propositions, are kept out of pr's domain.]

This problem has indeed been addressed, for the case of results and Hilbert space, in the only two papers I have found that apply the distinctions between our three propositions to a physical theory's probability ascriptions. Namely, van Fraassen and Hooker (1976, pp. 229–231) propose to solve the problem for this cae by using Popper functions; and Halpin (1991, p. 46) endorses this. I believe this proposal has some disadvantages, e.g., it cannot satisfy my constraint on values. But space prevents my showing this here (again, details available on request).

The second point is that my main conclusion—viz. we can construe physical theories' probability ascriptions as (ProbCond) or (ProbCons), and for finitely many quantities, as (CondProb)—is not just a consequence of my constructions' interpretations, of $\rightarrow$ or of pr. In fact, my constructions will take $\rightarrow$ as a strict conditional, and as a counterfactual conditional [as in Stalnaker (1968) and Lewis (1973)]. As to probability, for most of the paper I just need pr as a map on propositions (sets of worlds) subject to the classical probability axioms. [However, the nonclassical nature of quantum, i.e., Born rule probabilities will show up, especially for (ProbCond).] Nor will I need to take a stand on controversies about the interpretation of probability. And these interpretations typically uphold the distinctions between these three propositions. [In fact, we can adapt a single simple example so as to show all these distinctions, for this range of interpretations of $\rightarrow$, and for various interpretations of pr (Butterfield, 1992).] As we shall see, the main conclusion in fact follows from the way in which both phase space and Hilbert space allow certain kinds of independence between which quantity is measured and what the value of the quantity, or the measurement result obtained, is.

Third, I should sketch how my project supplements previous work by van Fraassen and Hooker (1976) and Halpin (1991) about which of our three propositions best suits a theory's probability ascriptions. They only consider the case of probabilities of results, for quantum theory. Their motivation is of course the familiar interpretative tradition saying that quantum theory's probabilities must be taken as conditional on a measurement being performed (e.g., Dirac, 1958, p. 47; Messiah, 1966, p. 294). But physicists tend to be unaware of our three distinctions. So it is natural to ask, as these authors do, whether such distinctions can shed light on the interpretation of quantum theory.

These authors agree in favor of a certain type of (CondProb); namely, with Popper functions. But this agreement belies other differences between them, of both overall aim and technical detail. As to aims: van Fraassen and Hooker aim to explicate Bohr's views, not to give some uniquely correct "logical form" or "regimentation" for quantum theory's probability ascriptions; while Halpin does aim for such a form, for which he lists desiderata. I myself am skeptical that there is such a form; but considering various possible forms is of course still worthwhile, since we understand something better when we can look at it in different ways. Another difference of aim is that van Fraassen and Hooker confine themselves to maximal quantities; but Halpin considers all quantities.

And as to technical details: van Fraassen and Hooker, after endorsing (CondProb), go on to hold that (ProbCond), with $\rightarrow$ as the Stalnaker conditional, also explicates Bohr's views. They do this by rigorously building "toy" possible worlds, in terms of ordered pairs of the quantity measured and the result obtained; they then interpret $\rightarrow$ as a Stalnaker conditional on these worlds; and they define a probability function on these worlds in terms of their (CondProb)'s Popper probability function. On the other hand, Halpin does not build such toy worlds. Although he assumes a possible-worlds semantics, he takes these worlds as primitive, not constructed. And with this framework, he argues against both (ProbCond) and (ProbCons), for the Stalnaker conditional and for any $\rightarrow$ with a certain trio of properties. (The trio seems innocuous, and is satisfied by familiar alternatives to the Stalnaker conditional, such as the strict conditional and Lewis' counterfactual conditional.)

Set aside the overarching, unresolved dispute about whether there is a correct logical form for quantum theory's probability ascriptions. [As mentioned, I am skeptical; and accordingly I will not be endorsing Halpin's arguments against (ProbCond) and (ProbCons).] These two papers still leave some more specific and tractable matters open. The obvious one arises from the fact that van Fraassen and Hooker do not consider (ProbCons), while Halpin does not build worlds. So can we build worlds and probabilities on them, in van Fraassen and Hooker's kind of way, so as to get (ProbCons) construals of quantum theory's probability ascriptions?

In fact we can: the construction is trivial; and we need no restriction to maximal quantities. With an analogous construction, we can also build worlds and probabilities for (ProbCond) rather differently from van Fraassen and Hooker. This construction is simpler than theirs; and again there is no restriction to maximal quantities. As mentioned above, both constructions are simple enough that I will not need to be as formal as van Fraassen and Hooker: there is nothing to prove—one only needs to think of the Hilbert space formalism in a certain way.

2. NOTATION

I shall use the following notation

$a, b, \ldots$	quantities that are measured
$c, d, \ldots$	quantities, perhaps measured, perhaps not
$A, B, \ldots$	intrinsic values, or measurement results, for quantities a, b
$C, D, \ldots$	intrinsic values for quantities c, d
$M_1, \ldots, M_i, \ldots$	a set, perhaps empty, of comeasurable quantities

I form the corresponding propositions by using a prime. So a' is the proposition that quantity a is measured, etc. NB: The propositions A', B', C', D' specify the quantity concerned; so A' entails a', etc. [If we did not have this convention, our three kinds of proposition (ProbCond), etc., would get an unintended interpretation—about how a raw number A as value or result affected the probability of which quantity was measured.] I also use the standard logical notation:

$$w \models a' \text{: possible world } w \text{ makes true } a'$$

As discussed in Section 1, I take the conditional connective $\rightarrow$ to be either a strict conditional, defined in terms of a binary relation of accessibility among worlds, or a Stalnaker–Lewis counterfactual conditional, defined in terms of similarity among worlds. In fact, later sections will take accessibility to be, as usual, reflexive. And they will endorse the so-called Limit Assumption, that at each world, for each antecedent X, there is a set of most similar X-worlds; so that I say:

$$w \models X \rightarrow Y \quad \text{iff all the } X\text{-worlds most similar to } w \text{ are } Y\text{-worlds}$$

I can now write down the forms of (ProbCond), etc., that will concern us. First, consider values. We want to talk about the probability that a quantity has a value, given that a certain quantity, perhaps different from the first, is measured; or given that a certain combination of quantities is measured. So we are concerned with

$$\mathrm{pr}(a' \rightarrow C') = z; \qquad a' \rightarrow (\mathrm{pr}(C') = z)$$

and more generally for combinations of quantities:

$$\mathrm{pr}(M_i' \rightarrow C') = z; \qquad M_i' \rightarrow (\mathrm{pr}(C') = z)$$

For results, the situation is simpler. We want to talk about the probability that a quantity has a result, given that this quantity, or a combination including it, is measured. So we are concerned with

$$\mathrm{pr}(a' \to A') = z; \qquad a' \to (\mathrm{pr}(A') = z)$$

and more generally for combination of quantities, requiring that $a \in M_i$;

$$\mathrm{pr}(M_i' \to A') = z; \qquad M_i' \to (\mathrm{pr}(A') = z)$$

I turn to notation for Hilbert space. I write H for Hilbert space, with elements $v \in H$. Each v assigns probabilities to subspaces C, D of H, according to the Born rule: $\mathrm{pr}_v(C) := \|\mathrm{Proj}_c(v)\|^2$. For noncommuting subspaces C and D, the probabilities are nonclassical in the familiar sense that marginals are not recovered in the obvious way:

$$\mathrm{pr}_v(C) \neq \mathrm{pr}_v(C \cap D) + \mathrm{pr}_v(C \cap D^{\perp})$$

Finally, to help distinguish different notions of probability, I will use: pr_v only for Born rule probabilities; PR only for a probability measure on possible worlds; and pr for the general idea of probability, e.g., as it occurs in our three propositions (ProbCond), etc.

3. CONSTRAINTS

With this notation in hand, I can motivate the constraints reflecting the distinction between values and results that most of my constructions will satisfy.

As I see it, the basic idea of values is that they are intrinsic properties of the system. To analyze "intrinsic" is difficult: the idea is roughly "would be possessed independently of the state of other objects". But I will consider only independence of whether a quantity is measured, and if so, which one. So we get: the value of a quantity would be what it is, whether or not that quantity, indeed any quantity, is measured. And similarly, I propose, for probabilities of values. So the idea of my constraint for values is:

> the probability of a given quantity taking a given value is the same, whichever quantity is measured.

This independence can be made precise, using our propositions (ProbCond), etc. If we consider just whether or not the quantity a is measured, we get

for (ProbCond): $\mathrm{pr}(a' \to C') = \mathrm{pr}(\neg a' \to C')$
for (ProbCons): there is a z s.t. both $a' \to (\mathrm{pr}(C') = z)$ and $\neg a' \to (\mathrm{pr}(C') = z)$

But we should also consider the various possible combinations of measurements. Given the set of quantities $\{a, b, c, d, \ldots\}$, there is a set of comeasurable subsets $\{M_i\}$. So we get as constraints:

(ValProbCond) for all M_i, M_j: $\mathrm{pr}(M_i' \rightarrow C') = \mathrm{pr}(M_j' \rightarrow C')$
(ValProbCons) there is a z s.t. for all M_i: $M_i' \rightarrow (\mathrm{pr}(C') = z)$

Turning to results, these must of course *not* be independent of whether a quantity is measured. Recall from Section 2 that result A identifies that quantity a is measured; so A' entails a'. This makes the above constraints, once amended by putting A' for C', inappropriate for results. For example, consider (ProbCond) for results, with a strict or counterfactual $\rightarrow$. For this case, the point is that there are no $(\neg a' \;\&\; A')$-worlds. This implies that $\mathrm{pr}(\neg a' \rightarrow A') = 0$; while we of course want $\mathrm{pr}(a' \rightarrow A')$ to equal what is prescribed by the state space. It is easy to check that similar remarks apply to (ProbCons).

But although results are not independent of what is measured, it is reasonable to require that their probability, as ascribed by the theory, is thus independent. In philosophical terms, the probability represents the strength of a disposition which is itself intrinsic to the system. So the idea of my constraint for results is:

> the probability that a measurement of a given quantity would yield a given result is to be the same, whichever quantity, if any, is measured

Again, this independence (this "same, whichever") can be made precise in terms of our propositions (ProbCond), etc.

I propose to use worlds, as in (ProbCons), to express the independence. So we get: across a class of worlds varying among themselves in what is measured, the probability, the strength of the disposition to give a certain result, is to be the same. For this to be so, the worlds must of course agree in their state vector $v \in H$. Assuming such a restriction on the worlds, and going directly to the case of combinations of measurements, I therefore propose the following constraints for results:

(ResProbCond) $\exists z$ s.t. at all worlds, & all M_i containing a: $\mathrm{pr}(M_i' \rightarrow A') = z$
(ResProbCons) $\exists z$ s.t. at all worlds, & all M_i containing a: $M_i' \rightarrow \mathrm{pr}(A') = z$

Finally, I should emphasize that I interpret these constraints, both for values and results, so as to allow the process of measurement to disturb values and/or results. I agree that at first glance, they seem not to allow this. Thus, if measurement disturbs values, then the probability of values for the time after the measurement will in general vary according to which quantity has just been measured, which seems to vitiate my constraint on values (in either of its forms). Similarly for results. A measurement might disturb the probabilities for results of a subsequent measurement in such a way that the probabilities vary according to which quantity has just been measured, which seems to vitiate my constraint on results (in either of its forms).

My reply is to clarify the role of time in the constraints. The argument just given reads the constraints as considering probabilities of values "for the time after the measurement" and probabilities for results "of a subsequent measurement." But I stipulate that in my constraints, X and Y are to be about the same time. That is, my constraint for values is [before formalization by (ProbCond), etc.]:

> the probability of a given quantity taking a given value at time t is (fixed by the system's state at t and so is) the same, whichever quantity, if any, is measured at t

And similarly my constraint for results is [before formalization by (ProbCond), etc.]:

> the probability that a measurement (on the system in its state at t) of a given quantity would yield a given result is (fixed by the state at t and so is) the same, whichever quantity, if any, is measured at t

This "same time" requirement applies throughout what follows. [Thus understood, these constraints can be applied to the ontological interpretation (Bohm, 1952) and to dynamical reduction models (Ghirardi *et al.*, 1986; Pearle, 1989), both of which explicitly model how measurement disturbs. But for reasons of space, I cannot discuss this.]

4. VALUES: (PROBCOND)

I only have space to consider what I take to be the orthodox view of quantum theory, namely that:

(i) The state of a system is fully described by a vector v (strictly, a ray) in the Hilbert space H.

(ii) In a state $v \in H$, the quantities that have values are exactly those for which v is an eigenvector (call these v's eigenquantities), and the value is the eigenvalue (for simplicity, I set aside the point that probability 1 does not imply certainty, so that perhaps even an eigenquantity can lack a value; but in later sections, I will respect this point, as it applies to results);

(iii) The result of a measurement of a is not determined by v, nor by v and a specification of other quantities measured together with a.

In this section and later, we will see a conflict between the nonclassical nature of quantum probabilities and classical probability measures on worlds. In particular, the constraints for (ProbCond), both for values and results, will only be satisfied in a restricted way. To that extent, (ProbCons) will be favored.

So first, the (ProbCond) construction for values. A world is to specify a state $v \in H$, the ensuing values, using the orthodox rule (ii) above, and which quantities are measured. I represent the ascription of values for any quantity c by a function $[c]$ sending states to values. So by (ii), this function has as domain the eigenvectors of c; or alternatively, it is defined on all of H, but takes a null value, say ∞, on all but eigenvectors of c. So I define a world w as a sequence containing a state v, and the quantities measured. I also write $s(w)$, the state of world w, for v. Then we have, for any c that has a value in $s(w)$,

$$w := \langle v, a, b, \ldots \rangle \qquad \text{and} \qquad w \models a', b', \text{ the value of } c = [c](s(w))$$

Now let W be a physical mixture of the v's; i.e., W is a density matrix with a physical decomposition into some, perhaps nonorthogonal, v's. It follows that if PR is a probability measure (of course classical) on worlds whose marginal on v matches W, then PR will match W's probabilities for eigenquantities c of the various v.

But of course there is a sense in which no classical PR on worlds can recover all the quantum probabilities prescribed by a vector $v \in H$. For recall from Section 2 that when quantities c and d do not commute, there are noncommuting eigenspaces, C and D say, such that the Born-rule probabilities $\mathrm{pr}_v(C)$, $\mathrm{pr}_v(D)$, $\mathrm{pr}_v(C \cap D)$, etc., violate classical probability.

Now I turn to defining $\rightarrow$. The situation is pleasingly simple. In this section and later, the definition is essentially the same for the strict or counterfactual conditional, for (ProbCond) for values. [It has to be different for (ProbCond) for results; cf. Section 6.] The definition will also be essentially the same for (ProbCons), both for values and results.

Namely, the $\rightarrow$ is to preserve the system's state, v. So for the strict $\rightarrow$, we define:

two worlds are accessible iff they have the same state

And for the counterfactual $\rightarrow$, we define, for any world w, the most similar (closest) a'-worlds to be

all worlds w', such that both $s(w') = s(w)$ and w' makes true a'

Furthermore, the situation allows us some choice. It is easy to check that the constructions in this and all later sections go through unaltered if we use either or both of the following strengthenings of accessibility and similarity:

(α) at any world w, the accessible/the closest a'-worlds match w as regards which quantities are measured: insofar as this is possible, if some sets of quantities are not comeasurable; and insofar as this is possible, since in these worlds a itself must be measured.

(β) (For similarity only: "centering"): At any world w, the closest a'-world is w itself, if w makes a' true.

So, specializing again to the case of (ProbCond) for values, I now define $\rightarrow$, strict or counterfactual, with only preservation of the state, $s(w) := v$, counting towards accessibility or similarity. Then we have:

$$w \models a' \rightarrow C' \quad \text{iff} \quad w \models C' \quad \text{iff} \quad [\text{by (ii)}] \quad s(w) \text{ is a } C\text{-eigenvector of } c$$

Indeed, allowing for combinations of measurements

$$\begin{aligned} w \models M_i' \rightarrow C' \quad &\text{iff} \quad w \models M_j' \rightarrow C' \quad \text{iff} \quad w \models C' \\ &\text{iff} \quad s(w) \text{ is a } C\text{-eigenvector of } c \end{aligned}$$

(Intuitively, each v is like a conjunction of conditionals of the form $M_i' \rightarrow C'$ for arbitrary M_i and for the C such that v is a C-eigenvector of c.)

It follows that any PR on the worlds whose marginal on v matches a physical mixture W will satisfy our constraint (ValProbCond) for the eigenquantities of the v's in the physical decomposition of W. Writing Proj_c for the projector onto eigenspace C of such an eigenquantity, we have

$$\mathrm{PR}(M_i' \rightarrow C') = \mathrm{PR}(M_j' \rightarrow C') = \mathrm{tr}(W \cdot \mathrm{Proj}_c)$$

But although (ValProbCond) is thus satisfied for these eigenquantities, note that these PR values only match some of the quantum probabilities: no classical PR on worlds can recover all the $\mathrm{pr}_v(C)$ as $\mathrm{PR}(a' \rightarrow C')$ or $\mathrm{PR}(M_i \rightarrow C')$.

Finally note that the nonuniqueness of PR reflects the fact that the construction is not committed to probabilities for a', b' (to a', b' being in the domain of PR). For the construction to be committed to such probabilities, they would have to be uniquely induced by probabilities it is committed to, i.e., by v.

5. VALUES: (PROBCONS)

If the probabilistic consequent in (ProbCons) is about values, we need probability ascriptions for values to be true at worlds—so that we get a proposition as second argument for the operation $\rightarrow$. So in this section, I take the Born rule probabilities to be about values, not results. So a world w is to specify a vector $v \in H$, which is the world's state $s(w)$, and which quantities are measured. But it must also make true probability ascriptions for values. So

$$w = \langle v =: s(w), a, b, \ldots \rangle \qquad \text{and} \qquad w \models a', b', \mathrm{pr}(C') = \mathrm{pr}_{s(w)}(C)$$

where $\mathrm{pr}_{s(w)}(C)$ is the Born rule probability of eigenspace C. As emphasized in Section 4, no classical PR on worlds can recover these quantum probabilities $\mathrm{pr}_{s(w)}(C)$ for all C.

Now define $\rightarrow$, strict or counterfactual, with only preservation of the state, $s(w)$, counting toward accessibility or similarity. Then for any world

$$w \models a' \rightarrow [\mathrm{pr}(C') = \mathrm{pr}_{s(w)}(C)], M_i' \rightarrow [\mathrm{pr}(C') = \mathrm{pr}_{s(w)}(C)]$$

The constraint (ValProbCons) is clearly satisfied. At the worlds with a common state, $\mathrm{pr}_{s(w)}(C)$ is the common value z such that for all M_i

$$M_i' \rightarrow [\mathrm{pr}(C') = z]$$

Finally, note that we could make an alternative construction. Namely, we could require also that each world make true nonprobabilistic ascriptions of values, for its states eigenquantities, according to (ii) of Section 4. Thus if $s(w)$ is a C-eigenvector of c, we get

$$w \models C' \qquad \text{and} \qquad w \models M_i' \rightarrow C'$$

in addition to what we already have,

$$w \models a' \rightarrow [\mathrm{pr}(C') = \mathrm{pr}_{s(w)}(C)]$$

This alternative in effect embeds Section 4's construction in that of this section.

6. RESULTS: (PROBCOND)

I turn to results in Hilbert space, maintaining the orthodox view announced at the start of Section 4. We will see that, as happened for values, (ProbCond) is less natural than (ProbCons). Indeed, it is less natural than it was for values: even to get satisfaction of (ResProbCond) for the eigenquantities of the state, we will need to revise our definition of $\rightarrow$ from the usual "preserve only the state $s(w)$." [This will also make contact with van Fraassen and Hooker (1976) and Halpin (1991), who, as discussed in Section 1, confine themselves to the case of results in Hilbert space.]

For (ProbCond), a world w is to specify a state $s(w) := v \in H$, which quantities are measured, and the results obtained.

Recall that a result specifies its quantity: in terms of propositions, A' entails a'. And in accordance with (iii) of Section 4, a result A is not determined by the state v, not even when taken together with the specification of other quantities measured together with a. Indeed, since probability 1 does not imply certainty (probability 0 events can happen), the result A is not determined even in the case where v is an A-eigenvector of a. So results for quantities cannot be represented by functions on H, even if parametrized by comeasurable sets, e.g., $[a_{Mi}]$. So we have

$$w = \langle v =: s(w), a, b, \ldots, A, B, \ldots \rangle \qquad \text{and} \qquad w \vDash a', b', A', B'$$

[We could require that w also make true value ascriptions for unmeasured quantities; according to (ii) of Section 4, for eigenquantities of $s(w)$. But as before, this would just embed a previous construction in the present one. For simplicity I omit it.]

We can put the above points about results in terms of an entailment and a nonentailment at the metalevel. A' entailing a' gives an entailment

$$w \vDash A' \quad \text{entails} \quad w \vDash a'$$

But the fact that A is not determined by $\langle v, a, b, \ldots \rangle$, even when v is an A-eigenvector of a, means that we have two nonentailments:

$$[w \vDash A'] \nRightarrow, \nLeftarrow [w \vDash a' \ \& \ s(w) \text{ is an } A\text{-eigenvector of } a]$$

Now suppose we define $\rightarrow$, strict or counterfactual, as usual, with only preservation of the state $s(w)$ counting toward accessibility or similarity [if one likes, with Section 4's strengthenings (α) and (β)]. The nonentailments above imply that the a'-worlds with a given state $s(w)$—the worlds to which we are led in a conditional with antecedent a'— vary in their results. They do so even if $s(w)$ is an A-eigenvector of a. So no world makes true the conditional $a' \rightarrow A'$ (for any result A). So any PR on the worlds must assign this conditional 0: $\text{PR}(a' \rightarrow A') = 0$. So no such PR recovers quantum theory's probability ascriptions as (ProbCond)s, even in some limited way. [Van Fraassen and Hooker make this point (1976, p. 226(a), p. 236); cf. also Halpin (1991, p. 47).]

However, there is a way to amend the definition of $\rightarrow$ so as to get a (ProbCond) recovery of the quantum probabilities for eigenquantities of the state, much like Section 4, and so as to get a correspondingly restricted satisfaction of the constraint (ResProbCond). Indeed, there are various ways one can amend the definition of $\rightarrow$ so as to get these conclusions. What matters is just that the definition has this property:

$$w \vDash a' \rightarrow A' \qquad \text{iff} \quad s(w) \text{ is an } A\text{-eigenvector of } a$$

For this will imply that for any classical measure PR on the worlds

$$\mathrm{PR}(a' \rightarrow A') = \mathrm{PR}(\text{the state is an } A\text{-eigenvector of } a)$$

As we shall see, this last easily delivers a restricted (ResProbCond).

So we could get what we want by just defining $\rightarrow$ by the property above. Another way is to follow Halpin's proposal (1991, pp. 53–55) [as mentioned in Section 1, he motivates it by rejecting (ProbCons)]. In our terminology, he proposes in effect that $\rightarrow$ should be understood as "would with quantum probability, prescribed by $s(w)$, equal to 1"; and that this is just the extreme, probability 1, case of a probabilistic counterfactual "would with probability p." In more detail; he assumes that for each world w and antecedent a', there is a set of most-similar-to-w a'-worlds (p. 42). Then he points out that at any world w, $s(w)$ defines a classical probability measure on result propositions A' (strictly speaking, on the intersections of these result propositions with w's set of closest a'-worlds). He then proposes that this probability measure gives the truth conditions of his probabilistic counterfactual. It follows immediately that if we read $\rightarrow$ in $a' \rightarrow A'$ as Halpin's "would with probability 1," then we get the property we want:

$$w \models a' \rightarrow A' \qquad \text{iff} \quad s(w) \text{ is an } A\text{-eigenvector of } a$$

Van Fraassen and Hooker give a technically more difficult amendment of $\rightarrow$, also aimed at (ProbCond) recovering quantum probabilities; they use supervaluations (van Fraassen and Hooker, 1976, pp. 237f), but neither they nor Halpin consider (ResProbCond), to which I now turn.

This property implies that for any classical measure PR on the worlds

$$\mathrm{PR}(a' \rightarrow A') = \mathrm{PR}(\text{the state is an } A\text{-eigenvector of } a)$$

For a fixed v, $\mathrm{PR}(a' \rightarrow A')$ will then be 0 or 1 according as v is or is not such an eigenvector. But $\mathrm{PR}(a' \rightarrow A')$ can be nontrivial (i.e., $\neq 0$ or 1) once we consider mixtures. Thus, let W be a physical mixture of the v's. Then, as in Section 4, if PR, a classical probability measure on the worlds, has a marginal on v that matches W, PR's assignment to conditionals $a' \rightarrow A'$, where a is an eigenquantity of one of the v's, will equal W's assignment.

$$\mathrm{PR}(a' \rightarrow A') = \mathrm{tr}(W \cdot \mathrm{Proj}_A), \qquad a \text{ an eigenquantity of one of the } v\text{'s}$$

And similarly for combinations of measurements M_i:

$$\mathrm{PR}(M_i' \rightarrow A') = \mathrm{tr}(W \cdot \mathrm{Proj}_A), \qquad a \in M_i, \text{ an eigenquantity of one of the } v\text{'s.}$$

Now we can get satisfaction of (ResProbCond), though only for the eigenquantities of the v's. It suffices to take this equation to be itself true at each world. If we do this, then the r.h.s, $\mathrm{tr}(W \cdot \mathrm{Proj}_A)$, is the common value z such that for all M_i containing a, it is true at each world that $\mathrm{PR}(M_i' \to A') = z$.

7. RESULTS: (PROBCONS)

In this final section, we shall see how (ProbCons) for results need only deal with classical probabilities and can satisfy the constraint (ResProbCons).

We begin much as in Section 6. A world is to specify all that it specified there, and also probabilities of results. To be precise, a world w is to specify a state $v \in H$, which quantities $a, b, \ldots$ are measured, Born rule probabilities of all these quantities' possible results (probabilities fixed by $\langle v, a, b, \ldots \rangle$, and the results actually obtained (not so fixed!). So we have

$$w = \langle v =: s(w), a, b, \ldots, A, B, \ldots \rangle$$

and

$$w \vDash a', b', A', B', \mathrm{pr}(A') = \mathrm{pr}_v(A), \ldots$$

where as usual $\mathrm{pr}_v(A)$ is the Born-rule probability. (As in Section 6, I could require that w also make true value ascriptions for unmeasured quantities; but as there, I omit this for simplicity.) So again, we have two nonentailments:

$$[w \vDash A'] \not\Rightarrow, \not\Leftarrow [w \vDash a' \ \& \ s(w) \text{ is an } A\text{-eigenvector of } a]$$

Even before defining $\to$, we can now see how treating results rather than values leads to a restriction to comeasurable sets and so to the adequacy of classical probability. Note that each state $s(w)$ fixes a measure, $\mathrm{PR}_{s(w)}$ say, on all the a'-worlds with state $s(w)$ by

$$\mathrm{PR}_{s(w)}(A') := \mathrm{pr}_{s(w)}(A) \qquad (:= \text{the Born-rule probability})$$

This generalizes to combinations of measurements. The set of all the worlds is partitioned by propositions stating which v is the state (we could write these propositions as v'). Any cell of this partition is itself partitioned into subcells by the propositions M_i'. In each subcell, the state v prescribes a wholly classical probability measure, $\mathrm{PR}_{v,i}$ say, on all propositions reporting single and joint results for quantities in M_i. (Strictly speaking, the domain is the intersections of these propositions with the subcell in question; but I shall not put this intersection in the notation.) Thus for pairs of results A, B taken as subspaces of H, we have

$$\mathrm{PR}_{v,i}(A' \ \& \ B') := \mathrm{pr}_v(A \cap B) \qquad (:= \text{the Born-rule probability})$$

Thus, treating results leads to a restriction to comeasurable sets, and so to the adequacy of classical probability (i.e., joint probabilities for all subsets of the comeasurable set).

Now define $\rightarrow$, strict or counterfactual, as usual, with only preservation of the state $s(w)$ counting toward accessibility or similarity [if one likes, with Section 4's strengthenings (α) and (β)]. Then for all worlds w and all M_i containing a, we have

$$w \models a' \rightarrow [\mathrm{pr}(A') = \mathrm{pr}_{s(w)}(A)],\ M_i' \rightarrow [\mathrm{pr}(A') = \mathrm{pr}_{s(w)}(A)]$$

where $\mathrm{pr}_{s(w)}(A)$ is as usual the Born-rule probability. Similarly for any joint probabilities of results for quantities $a, b, \ldots \in M_i$.

(ResProbCons) is satisfied: $\mathrm{pr}_{s(w)}(A)$ is the common value z such that for all M_i containing a, it is true at all worlds with state $s(w)$ that $M_i' \rightarrow [\mathrm{pr}(A') = z]$. And similarly for joint probabilities.

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